Qualifying Exam Review Notes

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1 Vector Calculus

1.1 Line Integrals

Two-dimensional line integrals, if a parametrization of the curve C exists in a function of one variable, takes the form

$$\oint_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{(\dot{x})^2+(\dot{y})^2}dt$$

Similarly, if we are looking at integrals just against dx and dy, we have the relation

$$\begin{cases} \int_C f(x,y)dx = \int_a^b f(x(t),y(t))\dot{x}(t)dt \\ \int_C f(x,y)dy = \int_a^b f(x(t),y(t))\dot{y}(t)dt \end{cases}$$

The next result follows from the Fundamental Thm of Calculus.

Theorem. *The Fundamental Thm for Line Integrals*

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient ∇f is continuous on C. Then

$$\oint_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Theorem. *Independence of Path*

$\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D iff $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem. *Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f s.t. $\nabla f = \mathbf{F}$.*

A conservative vector field $\mathbf{F}$ is such that $\nabla \times \mathbf{F} = 0$.

1.2 Green's Theorem

Theorem. *Green's Thm*

Let C be a positively-oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

The vector form of Green's Thm takes the form

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}ds = \int\int_D \nabla\cdot\mathbf{F}(x,y)dA$$

Theorem. *If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ with continuous second-order partial derivatives, then*

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0$$

1.3 Surface Integrals

Surface integrals: over a surface S, region bounded by S is D.

$$\int\int_S f(x,y,z)dS = \int\int_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}dA$$

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$$

Theorem. 1.4 Stoke's Theorem

Stoke's Thm

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$$

1.5 The Divergence Theorem

Theorem. *The Divergence Thm*

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int\int_E \nabla\cdot\mathbf{F}dV$$

1.6 Useful Vector Identities

Note: All relations only hold when vector field is well-defined (smooth). The function f is smooth, scalar.

$$\begin{aligned}
\nabla\times(\nabla\times A) &= \nabla(\nabla\cdot A)-\nabla^2 A\\
\nabla\cdot(A\times B) &= B\cdot\nabla A - A\cdot\nabla\times B\\
\nabla\times(A\times B) &= A(\nabla\cdot B)-B(\nabla\cdot A)\\
&+ (B\cdot\nabla)A-(A\cdot\nabla)B\\
\nabla\cdot(\nabla\times A) &= 0\\
\nabla\times(fA) &= f(\nabla\times A)+(\nabla f)\times A
\end{aligned}$$

2 Linear Algebra

2.1 Useful Information

A complex matrix is **unitary** if $A^* = A^{-1}$.

A complex matrix is **normal** if $AA^* = A^*A$. A Hermitian $\implies$ A normal.

Theorem. *Let A be a square matrix. Then the following are equivalent:*

1. *A is invertible (non-singular)*
2. *A is row equivalent to the identity matrix I*
3. *A is a product of elementary matrices*

Definition. Let $F : V \to U$ be a linear a mapping. The **kernel** is the set of elements in V that map into the zero vector 0 in U; that is,

$$\text{Ker}F = \{v \in V : F(v) = 0\}$$

The **image** (or **range**) of F, is the set of image points in U; that is,

$$\text{Im}F = \{u \in U : \text{ there exists } v \in V \text{ for which } F(v) = u\}$$

Proposition. *Let A be any $m\times n$ matrix over a field K viewed as a linear map $A : K^n \to K^m$. Then*

$$\mathit{KerA} = \mathit{nullsp}(A) \text{ and } \mathit{ImA} = \mathit{colsp}(A)$$

Theorem. *Let V of finite dimension, and let $F : V \to U$ be linear. Then*

$$\mathit{dimV} = \mathit{dim}(\mathit{KerF}) + \mathit{dim}(\mathit{ImF}) = \mathit{nullity}(F) + \mathit{rank}(F)$$

2.2 Gram-Schmidt Orthogonalization Process

Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis $\{w_1, w_2, \ldots, w_n\}$. Set

$$\begin{aligned}
w_1 &= v_1\\
w_2 &= v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >}w_1\\
&\vdots\\
w_n &= v_n - \sum_{i=1}^{n-1}\frac{< v_n, w_i >}{< w_i, w_i >}w_i
\end{aligned}$$

2.3 Determinants

Let $A, B \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}$, A invertible

$$\begin{aligned}\det(AB) &= \det(A)\det(B) = \det(BA)\\ \det(cA) &= c^n\det(A)\\ \det(A^{-1}) &= (\det(A))^{-1}\\ \det(A^T) &= \det(A)\end{aligned}$$

2.4 Diagonalizability

Suppose an n-square matrix A is given. The matrix A is said to be **diagonalizable** if there exists a non-singular matrix P s.t.

$$B = P^{-1}AP$$

is diagonal.

Theorem. *An n-square matrix A is similar to a diagonal matrix D iff A has n linearly independent eigenvectors. in this case, the diagonal elements of D are the corresponding eigenvalues and $D = P^{-1}AP$, where P is the matrix whose columns are the eigenvectors.*

Theorem. *Let A be a real symmetric matrix. Then there exists an orthogonal matrix P s.t. $D = P^{-1}AP$ is diagonal.*

2.5 Linear Operators on Inner Product Spaces

A linear operator T on an inner product space V is said to have an **adjoint operator** T^* on V if $< T(u), v >=< u, T^*(v) >$ for every $u, v \in V$.

Theorem. *Let T be a linear operator on a finite-dimensional inner-product space V over K. Then*

1. *There exists a unique linear operator T^* on V s.t. $< T(u), v >=< u, T^*(v) >$ for every $u, v \in V$. (That is, T has an adjoint T^*.)*
2. *If A is the matrix representation T wrt any orthonormal basis $S = \{u_i\}$ of V, then the matrix representation of T^* in the basis S is the conjugate transpose A^* of A (or the transpose A^T of A when K is real).*

3 Analysis

3.1 Metric Spaces

Definition. A metric d on a space M satisfies the following:

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)\ \forall x, y \in M$
3. $d(x, z) \le d(x, y) + d(y, z)$

with $d : M \times M \mapsto [0, \infty)$.

$l^p(\mathbb{R}, \mathbb{N})$ with $||x||_p = (\sum_{\mathbb{N}} |x_i|^p)^{(1/p)}$ is a metric space. In particular $l^1 \subset l^2 \subset \cdots \subset l^\infty$.

Convergence in metric space: If $x_n \to x$ in (M, d) then $\exists\epsilon > 0$, $\exists N_\epsilon$ s.t. $n > N_\epsilon \implies d(x_n, x) < \epsilon$.

Cauchy Convergence: A sequence x_n is Cauchy iff the following holds: $\forall\epsilon > 0$ there exists N_ϵ s.t.

$$n, m \ge N_\epsilon \implies d(x_n, x_m) < \epsilon$$

In particular, if all Cauchy sequences in M converge to a limit in M, the space is *complete*. Additionally, all convergent sequences are Cauchy.

Coarseness: If $x_n \to x$ in (M, d) implies that $x_n \to x$ in (M, d') then d is coarser than d'.

Equivalence of metrics: If two metrics are equivalent then

$$x_n \to x \text{ in } (M, d) \iff x_n \text{ in } (M, d')$$

E.g. On $\mathbb{R}^n$ all l^p norms are equivalent. An alternate characterization is the following: if given x and $\epsilon > 0$, $\delta > 0$ s.t.

$$d'(x, y) < \delta \implies d(x, y) < \epsilon$$

and $\exists\delta' > 0$

$$d(x, y) < \delta' \implies d'(x.y) < \epsilon.$$

Continuity in a metric space: Continuity of $f : V \mapsto W$ at v_0 can be defined through the following: $\forall\epsilon > 0$ there exists $\delta > 0$ s.t.

$$||v - v_0|| < \delta \implies ||f(v) - f(v_0)|| < \epsilon$$

This is implied by $||f(v)||_W \le C||v||_V$. Continuity can also be defined sequentially, i.e. if $x_n \to x$ and $f(x_n) \to f(x)$, f is continuous.

Hölder's inequality: For all sequences we have the following:

$$\sum_{\mathbb{N}} |x_i y_j| \leq ||x||_p ||y||_q$$

if $x \in l^p$ and $y \in l^q$ with $1/p + 1/q = 1$. For integrals, $f \in L^p$, $g \in L^q$,

$$\int_{\mathbb{R}} |f(x)g(x)|dx \leq ||f||_{L^p} ||g||_{L^q}$$

Young's inequality

$$ab \leq a^p/p + b^q/q$$

or more generally:

$$ab \leq \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt$$

These are used to prove **Minkōwski's inequality**:

$$\left(\sum_{n=1}^{\infty}(x_n + y_n)^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} x_n^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p\right)^{1/p}$$

or

$$\begin{aligned}\left(\int_a^b (f(x) + g(x))^p dx\right)^{1/p} &\leq \left(\int_a^b f(x)^p\right)^{1/p} \\ &+ \left(\int_a^b g(x)^p\right)^{1/p}\end{aligned}$$

if $\{x_n\}, \{y_n\} \in l^p(\mathbb{N})$ and $f, g \in L^p\big([a, b]\big)$.

3.2 Normed linear spaces = vector spaces

Definition. A vector space V on a field F (usually $\mathbb{R}$) is defined via the following; first the properties of the vector space ($u, v, w \in V$):

1. $v + w = w + v$
2. $u + (v + w) = (u + v) + w$
3. $\exists 0 \in V$ s.t. $0 + u = u$
4. $\exists -v \in V$ s.t. $-v + v = 0$

and operations involving elements $a, b \in F$:

1. $a(v + w) = av + aw$
2. $(a + b)v = av + bv$
3. $a(bv) = b(av)$
4. $1 \cdot v = v$ where 1 is the multiplicative identity of F

The norm on this space satisfies

- $||v|| \geq 0$
- $||av|| = |a|||v||$
- $||v + w|| \leq ||v|| + ||w||$

A metric is thus defined as $d(x, y) = ||x-y||$, so every vector space is a metric space.

Continuity of linear maps: A linear map $f : (V, ||\cdot||_V) \mapsto (W, ||\cdot||_W)$ is defined with the following

1. $f(av + bw) = af(v) + bf(w)$
2. $f(0_V) = 0_W$

The following are equivalent for normed spaces V, W:

1. f is continuous
2. f is continuous at 0
3. $\exists C < \infty$ s.t. $||f(v)||_W \leq C||v||_V$

The space of bounded linear maps is denoted by $\mathcal{L}(V, W)$ and we have the following

1. If $(W, ||\cdot||_W)$ is complete, $\mathcal{L}(V, W)$ is complete.
2. Define the induced norm on this space as
$$||f||_{\mathcal{L}(V,W)} = \sup_{v \in V} \frac{||f(v)||_W}{||v||_V}$$
3. Let $||\cdot||$ denote $||\cdot||_{\mathcal{L}(V,W)}$, then
$$||AB|| \leq ||A||||B||$$

E.g. For matrices, we can find from the above construction that

1. $||A||_1 = \max_i \sum_j |a_{ij}|$
2. $||A||_2 = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximal eigenvalue of A^*A.
3. $||A||_\infty = \max_j \sum_i |a_{ij}|$

Dual Space V^*. This is the space is the one-to-one correspondence to the space of bounded linear maps $\mathcal{L}(V, \mathbb{R})$. I.e. $f \in \mathcal{L}(V, \mathbb{R}) \iff v \in V^*$.

Banach Spaces are vector spaces that are complete; i.e. all Cauchy sequences converge to a limit in the space itself. **E.g.** $C([a,b])$ with $||\cdot||_\infty$ is complete as well as C_0 with $||\cdot||_\infty$. Counterexamples include $C([a,b])$ with the L^2 or l_c (sequences with finite non-zero terms) with any l^p norm.

Semi-norms A semi-norm $p = L(v)$ for $v \in V$ and $L \in \mathcal{L}(V,\mathbb{R})$ satisfies

1. $p(av) = |a|p(v)$
2. $p(v+w) \leq p(v) + p(w)$

bf E.g. In particular we can build the l^1 norms on $\mathbb{R}^2$ through appropriate choices of p_1 and p_2 and take the supremum over semi-norms. I.e. define $p_1 = x+y$ and $p_2 = x-y$ to produce that

$$||(x,y)||_1 = \sup_{i=1,2} p_i(v)$$

For the l^2 norm we need an arbitrary collection $p_a = x\cos a + y \sin a$. Its supremum over all $a \in [0, 2\pi]$ gives the 2-norm on vectors. In general, we can build norms out of semi-norms when $\forall v \neq \implies p(v) > 0$. This mechanism we use is summing over many norms or taking a supremum.

3.3 Hilbert Spaces

Definition. Normed linear spaces that are complete. The norm must satisfy the parallelogram law:

$$||x+y||^2 + ||x-y||^2 \leq 2\big(||x||^2 + ||y||^2\big)$$

Cauchy-Schwartz says

$$| < x, y > | \leq ||x||_H ||y||_H$$

For M a closed linear space we can show that

$$M \oplus M^\perp = H$$

so we can decompose any $z \in H$ as $z = u + v$ for $u \in M$, $v \in M^\perp$.

Also, (for convex subspace as well), we can find unique $w \in M$ s.t.

$$\min_{v \in M} ||z - v|| = ||w - z||, \; w \in M$$

3.4 Topological Spaces

Definition. The following axioms define a topology $\mathcal{T}$ on a set X which is a collection of *open sets*,

1. $\emptyset, X \in \mathcal{T}$

The **relative topology** is the topology that is restricted to a subset $Y \subset X$ s.t.

$$\mathcal{S} = \{H \subset Y | H = G \cap Y, G \in \mathcal{T}\}$$

A **Hausdorff** topology is s.t. for all $x, y \in X$ there exists open sets $U \ni x, V \ni y$ s.t. $U \cap V = \emptyset$.

A subset $V \subset X$ is a *neighborhood* of $x \in X$ if V is open and $V \ni x$.

Convergence in topological sense is defined through the following: $x_n \to x$ iff $\forall V \in \mathcal{N}(x)$ there exists N s.t.

$$n \geq N \implies x_n \in V$$

where $\mathcal{N}(x)$ is a local base at x.

Continuity at a point is given by the following: for $f : X \mapsto Y$ then for f to be continuous at $x \in X$, then $\forall W \in \mathcal{N}(f(x)), \exists V \in \mathcal{N}(x)$ s.t. $f(V) \subseteq W$.

Continuity of functions in the topological sense is implied by the following: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. $f : X \mapsto Y$ is continuous iff $f^{-1}(G) \in \mathcal{T}$ for all $G \in \mathcal{S}$. Note that sequential continuity does not directly imply the above. I.e. the co-countable topology on $\mathbb{R}$.

A **Homeomorphism** is a function that is one-to-one and onto from one topological space to another.

A **Neighborhood** or **Local Base** is defined by the following axioms:

1. $V \in \mathcal{N}(x) \implies x \in V$
2. If $U, V \in \mathcal{N}(x)$, then $\exists W \in \mathcal{N}(x)$ s.t. $W \subset V \cap U$
3. If $V \in \mathcal{N}(x)$ there exists $W \in \mathcal{N}(x)$ s.t.
   (a) $W \subset V$
   (b) $y \in W \implies \exists U \in \mathcal{N}(y)$ and $U \subset V$.

We can define a topology using the local base; i.e. O is open if $\forall x \in O$ there exists $V \in \mathcal{N}(x)$ s.t. $V \subseteq O$.

A **base** is defined through the following axioms for $B \in \mathcal{B}$:

- $\bigcup_{B \in \mathcal{B}} B = X$
- For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $(B_3 \ni x) \in \mathcal{B}$ s.t. $B_3 \subseteq B_1 \cap B_2$.

Additionally, we can define a unique topology $\mathcal{T}$ on X with the base defined above

$$O \in \mathcal{T} \implies \forall y \in O, \exists B \in \mathcal{B} \text{ s.t. } (B \ni x) \subseteq O$$

In particular, we can thus find that for $O \in \mathcal{T}$ then $O = \bigcup_{U \in \mathcal{T}, U \subseteq O} U$.

The **weak topology** on X is defined as the topology that makes all linear functionals continuous; i.e. $f \in \mathcal{L}(V, \mathbb{R})$ is continuous.

A topological space is **first countable** if it has a countable local base. A topological space is **second countable** if it has a countable base. A space is second countable iff it is **separable**, which implies that there is a countable set $\{x_n\} \subset M$ s.t. given any $x \in M$ and $\epsilon > 0$ then $\exists x_n$ s.t. $d(x_n, x) < \epsilon$.

3.5 Size of Sets

3.5.1 Small/Largeness

A set A is **dense** in X if $U \cap A \neq \emptyset$ for all open $U \subset X$.

$A \subset X$, a point $x \in A$ is an **interior point** if $\exists U \ni x$ open s.t. $U \subset A$. Therefore

$$A^o = \{x | x \in \text{ interior of } A\} = \bigcap_{V^{\text{open}} \subset A} V$$

is called the **interior** of A.

For $A \subset X$, $x \in \bar{A}$ if for all open $V \ni x$, $V \cap A \neq \emptyset$, or

$$\bar{A} = \bigcup_{A \subseteq F, F \text{ closed}} F$$

The set $\bar{A}$ is known as the **closure** of A.

Note: $A^o \subset A \subset \bar{A}$. Clearly, A^o is open, $\bar{A}$ is closed.

$A \subset X$ is **nowhere dense** if (1) $\bar{A}$ has empty interior, i.e. $(\bar{A})^o = \emptyset$ or (2) $\overline{(A^c)^o} = X$.

If A is the countable union of nowhere dense sets, then A is **meager** or **of 1^st^ category**. I.e. $\{r_n\}$ is an enumeration of the rationals and so, since $\mathbb{Q}$ is a countable union of singleton sets, and so $\{r_n\}$ is meager. A single element set is nowhere dense.

The complement of a meager set is residual.

Density, again If $A \subset X$ is dense in X then $\bar{A} = X$. Also, if $A \subset X$ is dense in B, then $B \subset \bar{A}$. Also, A is dense in X iff its complement has empty interior.

very small	nowhere dense
very large	dense interior
small	meager
large	residual
medium small	empty interior
medium large	dense

A subset A of $\mathbb{R}^n$ has **measure zero** (or **Lebesgue measure zero**) if $\forall \epsilon > 0$, there is a countable collection $\{B_n\}$ of open balls $B_n = B(x_n, \delta_n) = \{x | \, ||x - x_n||_2 < \delta_n\}$ s.t.

$$A = \bigcup_{n=1}^{\infty} B_n$$

and $\sum_{n=1}^{\infty} \text{vol}(B_n) \leq \epsilon$. Example: $\mathbb{Q}$ in $\mathbb{R}$. The Cantor middle thirds set has measure zero. Any countable union of singleton sets has measure zero.

3.5.2 Measure theory

A **measurable space** is a set X together with a collection of subsets $\mathcal{B}$ of X satisfying:

- $X \in \mathcal{B}, \emptyset \in \mathcal{B}$
- $A \in \mathcal{B} \implies A^c \in \mathcal{B}$
- $A_1, A_2 \dots \in \mathcal{B} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$

$\mathcal{B}$ is called a σ-algebra. $\mathcal{B}$ is also closed under countable intersections.

A **measure** μ satisfies the following relations on a measurable space $(X, \mathcal{B})$,

- $\mu : \mathcal{B} \to [0, \infty]$
- $\mu(\emptyset) = 0$
- for A_n disjoint, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

More generally, for any collection of sets B_n, $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n)$.

If $A \subset B$, then $\mu(A) < \mu(B)$.

If $A \subset B$, then $B = (B \setminus A) \cup A$ and $\mu(B) = \mu(B \setminus A) + \mu(A)$.

If $A_n \in \mathcal{B}$ and $A_1 \supset A_2 \supset \dots$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

The **Borel σ-algebra** is the smallest σ-algebra containing all open sets in a topology $\mathcal{T}$.

3.6 Convergence and Compactness

3.6.1 Convergence and Completeness

Convergence in a topological space for a sequence $x_n \to x$ is defined by: for each open set or neighborhood of $x \in V$, $\exists N$ s.t. $n > N \implies x_n \in V$.

Convergence in a metric space: Given a sequence $\{x_n\}$; for all $\epsilon > 0$, $\exists N_\epsilon$ s.t. $n > N_\epsilon \implies d(x_n, x) < \epsilon$.

A sequence is **Cauchy** if $\forall \epsilon > 0$, $\exists N_\epsilon$ s.t. $m, n \geq N_\epsilon \implies d(x_n, x_m) < \epsilon$.

A metric space is **complete** if every Cauchy sequence converges to a limit in the metric space. E.g., the rationals are incomplete.

Contraction mapping theorem: Given a complete metric space (M, d), let $f : M \mapsto M$ be a continuous function and suppose that $\exists k$ with $|k| < 1$ s.t.

$$d(f(x), f(z)) \leq kd(x, z) \ \forall x, z \in M$$

then there exists a unique $y \in M$ s.t. $f(y) = y$.

Baire Category Theorem: In a complete metric space, every residual set is dense.

Further, this implies that every complete metric space is *not* a countable union of disjoint nowhere dense sets.

3.6.2 Completions

An **equivalence relation** on X is a subset of $X \times X$, i.e. a subset of pairs $\{(x_1, x_2) | x_1, x_2 \in X\}$ or written $x_1 \sim x_2$ that satisfy:

1. $x_1 \sim x_1$
2. $x_1 \sim x_2 \implies x_2 \sim x_1$
3. $x_1 \sim x_2, x_2 \sim x_3 \implies x_1 \sim x_3$

An **equivalence class** of x (denoted $[x]$) is the set of points equivalent to it, i.e.

$$[x] = \{y | y \sim x\}$$

The **completion** of a possibly incomplete metric space (M, d) is defined in the following way: if $\{y_n\}$ and $\{x_n\}$ are two Cauchy sequences in M we define the equivalence relation as $\{x_n\} \sim \{y_n\}$ if $d(x_n, y_n) \to 0$ as $n \to \infty$. $\bar{M}$ is the collection of all equivalence classes of Cauchy sequences in M. For such a space, define the distance metric as $\bar{d}$,

$$\bar{d}([\{x_n\}], [\{z_n\}]) = \lim_{n \to \infty} d(x_n, z_n)$$

Therefore, $(\bar{M}, \bar{d})$ is a complete metric space. Additionally, M is dense in $\bar{M}$.

3.6.3 Compactness

The property of **compactness** is morally an approximation to the properties of finite sets. All finite element sets are compact. More generally we define a compact set on metric spaces according to the following definition: For a compact set $A \subset X$, any sequence in A has a convergent subsequence. The real numbers are not compact, consider $\{x_n\} = n$.

Facts:

- A compact metric space is complete.
- A complete metric space is not necessarily compact. I.e. $\mathbb{R}$.
- A compact subset is closed and bounded.

A **bounded** set is defined by the parallel constructions

- $\exists K > 0.$ s.t. $d(x.y) < K$, $\forall x, y \in A$
- $\exists x_0 \in A$, $P < \infty$ s.t. $d(x_0, y) < P$, $\forall y \in A$.

Heine-Borel Theorem A subset A of $\mathbb{R}^n$ is compact iff it is closed and bounded. A corollary states that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Continuous functions on compact sets

- For $f : M \mapsto \mathbb{R}$ continuous, and so $\exists x_\pm$ s.t. for all $x \in M$

$$a = f(x_-) \leq f(x) \leq f(x_+) = A$$

For a topological space, a set A is **compact** if every open cover of A has a finite subcover.

Facts:

- A closed subset of a compact metric space is compact.
- A compact subset of a Hausdorff space is closed.

3.6.4 Uniformity

Let $f : (M, d) \mapsto (M', d')$ then f is **uniformly continuous** if $\forall \epsilon > 0$, $\exists \delta > 0$, s.t.

$$d(x, y) < \delta \implies d'(f(x), f(y) < \epsilon$$

δ depends only on ϵ, and not x

A continuous function from a compact metric space is uniformly continuous on the image.

Pointwise Convergence: Let (M,d) and (M',d') be metric spaces, then $\{f_n\}$ area sequence of functions from M to M' and suppose that $f : M \mapsto M'$. If $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in M$, then f_n approaches f pointwise.

Uniform Convergence If $\forall\epsilon > 0$, $\exists N$ s.t. $n > N \implies d'(f_n(x), f(x)) < \epsilon\ \forall x \in M$.

Weierstrauss M-test: Suppose f_n is a sequence of functions in (M,d) and suppose that

$$||f_n||_\infty \le a_n \text{ and } ||a_n||_1 < \infty$$

then the partial sums converge uniformly for large enough m, that is

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly}$$

The **uniform limit of continuous functions is continuous!** It means that $C(M)$ is complete for (M,d) a metric space.

Term-by-term differentiation. Suppose $f(x) = \sum_{n=0}^{\infty} f_n(x)$ converges uniformly on some interval $J \subset \mathbb{R}$ and also the sum of the derivatives, $g(x) = \sum_{n=1}^{\infty} f_n'(x)$ converges uniformly as well, then $f'(x) = g(x)$.

Integration of a uniformly convergent sequence is allowed but can be relaxed (using Lebesgue integration). At present, knowing that $\{f_n\}$ is a uniformly convergent sequence on $[a,b]$ then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Now we turn to parametrized integrals. So we define uniform convergence for integrals, i.e.

$$\int_{\mathbb{R}} f(t,x)dx = F(t)$$

converges uniformly to $F(t)$ for $t \in [a,b] \subseteq \mathbb{R}$, if for every $\epsilon > 0$ there is an R_0 depending on ϵ but not t, s.t. for all $R > R_0$, $\forall t \in [a,b]$,

$$\left| \int_{-R}^{R} f(t,x)dx - F(t) \right| < \epsilon$$

Differentiating parametrized integrals, if both (1) $\int_{\mathbb{R}} f(t,x)dx = F(t)$ and (2) $\int_{\mathbb{R}} \frac{\partial f}{\partial t} dx = G(t)$ converge uniformly, then $F'(t) = G(t)$.

3.7 Lebesgue Integration

The motivation for developing the idea of Lebesgue integration comes from consideration of the completion of $C([0,1])$ under the L^1 metric for example. It turns out these are measurable functions and the metric is the Lebesgue integral.

Define $U(f,\Delta) = \sum_{i=0}^{n} \max_{x\in(t_i,t_{i+1})} f(t)(t_{i+1} - t_i)$ and $L(f,\Delta) = \sum_{i=0}^{n} \min_{x\in(t_i,t_{i+1})} f(t)(t_{i+1} - t_i)$. These are approximations to the integral/area under the curve of f. Therefore the limiting process of the process can be defined:

$$U(f) = \inf_{\Delta} U(f,\Delta),\ \ L(f) = \sup_{\Delta} L(f,\Delta)$$

In particular, if f is Riemann integrable then $U(f) = L(f)$ and so for any continuous function this condition is met since they will be Riemann integrable. Can also that a general function on $[0,1]$ is Riemann integrable iff the set of points of discontinuity of f has Lebesgue measure zero.

We define **measurable functions**, take $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be measurable spaces, then a function $f : X \mapsto Y$ is **measurable** if $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{C}$. If f, g are measurable functions, then so to are (1) $f+g$ and (2) fg.

Let L^+ denote the space of all measurable functions from X to $[0,\infty]$.

Theorem. *If $\{f_n\}$ is a finite or infinite sequence in L^+ and $f = \sum_n f_n$, then $\int f = \sum_n \int f_n$.*

Proposition. *If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.*

This definition can be extended to the case that really interests us; when $Y = \mathbb{R}$. Therefore, first define the extended real numbers:

$$\mathbb{R}_{\text{extended}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$$

with the following rules: $a \in \mathbb{R}$, $a \pm \infty = \pm\infty$, if $a > 0$, $a \cdot \pm\infty = \pm\infty$, $0 \cdot \pm\infty = 0$, and $\infty - \infty$ is undefined.

Consider $(X,\mathcal{B})$ a measurable space. Then $f : X \mapsto \mathbb{R}_{\text{extended}}$ is measurable if the set

$$\{t | f(t) < \alpha\} \in \mathcal{B},\ \alpha \in \mathbb{R}$$

This property is inherited to a sequence of such functions, i.e. if $\{f_n\}$ are measurable then if $f_n \to f$ pointwise, then f is measurable.

3.7.1 Lebesgue integral

Definition. The characteristic function is

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

A **nonnegative simple function**: Let $A_1, \dots, A_n \subset X$ be pairwise disjoint measurable sets, a function

$$\phi(x) = \sum_{j=1}^{n} \alpha_j \chi_{A_k}$$

$$\int \phi = \sum_{j=1}^{n} \alpha_j \mu(A_j)$$

The idea is to approximate the integral of a general function f with successive simple functions of increasing n.

Let $f : X \mapsto [0, \infty]$ and measurable, then the Lebesgue integral is given by

$$\int f = \sup_{0 \leq \phi, \phi \text{ simple}} \int \phi$$

For a general function which is negative, just take the integration over the sets where f has positive values and where it has negative values.

3.7.2 Convergence theorems

Fatou's lemma

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu$$

Monotone Convergence Theorem: If there is a sequence of non-negative measurable functions $\{f_n\}$ s.t. $f_{n+1}(x) \geq f_n$ for all x and the pointwise limit of f_n is f, then

$$\lim \int f_n d\mu = \int \lim f_n d\mu$$

Dominated Convergence Theorem: If a sequence $\{f_n\}$ can be dominated by an integrable function g ($|f_n| \leq g$) and $\lim f_n = f$ then

$$\lim \int f_n d\mu = \int \lim f_n d\mu$$

4 Numerical Analysis

4.1 Solution of Nonlinear Equations

4.1.1 Bisection Method

Algorithm

1. is $f(a)f(b) < 0$? then $c = \frac{1}{2}(a + b)$
2. is $f(a)f(c) < 0$?

   repeat

$$\begin{cases} \text{if } (f(a)f(c) < 0) \\ b = c \\ \text{else } a = c \\ \text{else } f(c) = 0 \end{cases}$$

In implementation, use 3 stopping criteria

1. maximum number of iterations
2. location: $|b - a| < \epsilon$
3. accuracy of root: $|f(c)| < \delta$

Theorem. *Bisection Method: If* $[a_0, b_0], [a_1, b_1], \dots, [a_n, b_n], \dots$ *denote the intervals in the bisection method, then the* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, are equal and represent zero of* f. *If* $f = \lim_{n\to\infty} \frac{1}{2}(a_n + b_n)$ *then* $|r - c_n| \leq 2^{-n+1}(b_0 - a_0)$.

Linear convergence.

4.1.2 Newton-Raphson Method

Algorithm

1. Set $i = 1$
2. while $i \leq N_0$ do steps 3-6
3. Set $p = p_0 - f(p_0)/f'(p_0)$
4. If $|p - p_0| < Tol$ then

   output p

   break
5. iterate $i = i + 1$
6. $p_0 = p$
7. Output ('Method failed after N_0 iterates')

   break

PROS: Quadratic convergence

CONS: (1) Need $f'(x)$ or an approximation to it, however an approximation may not converge quadratically. (2) Not guaranteed to always converge.

Theorem. *Let $f \in C^2[a,b]$. If $p \in [a,b]$. If $p \in [a,b]$ is a simple zero (s.t. $f(p) = 0$) and $f'(p) \neq 0 \implies \exists$ a neighborhood of p and a constant C s.t. if the Newton method started in that neighborhood, successive guesses become closer to p and satisfy*

$$|p_{n+1} - p| \leq C(p_n - p)^2$$

In some cases, the Newton method is guaranteed to converge from any arbitrary starting point.

Theorem. *If $f \in C^2(\mathbb{R})$, is an increasing, convex function, and $f(p) = 0$ then p is unique and the Newton method will converge to it from any starting point.*

4.1.3 Steffensen Method

Similar to Newton-Raphson, except no $f'(x)$ is not needed.

$$\begin{aligned} p_{n+1} &= p_n - f(p_n)/g(p_n) \\ \text{where } g(x) &= \frac{f\big(x + f(x)\big) - f(x)}{f(x)} \\ \text{This requires } g &\in C^3[a,b] \text{ for quadratic} \\ \text{convergence} & \end{aligned}$$

4.1.4 Secant Method

Like the Newton Method with a secant approximation to $f'(x)$. Replace $f'(x)$ with $[f(x_n) - f(x_{n-1})]/(x_n - x_{n-1}$. The iteration becomes

$$p_{n+1} = p_n - f(p_n)\left[\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}\right] \quad n \geq 1$$

Rate of Convergence: $e_{n+1} = \mathcal{C}e_n e_{n-1} \sim A|e_n|^{(1+\sqrt{5})/2}$

4.2 Useful Norm Information

For $A \in \mathbb{C}^{m\times n}$

$$\frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{m}||A||_\infty$$

4.3 Singular Value Decomposition

For $A \in \mathbb{C}^{m\times n}$

$$A = U\Sigma V^*$$

where

$$\begin{aligned} U &\in \mathbb{C}^{m\times m} \text{ is unitary,} \\ V &\in \mathbb{C}^{n\times n} \text{ is unitary,} \\ \Sigma &\in \mathbb{R}^{m\times n} \text{ is diagonal.} \end{aligned}$$

4.3.1 Change of Basis

If $b' = U^*b$, $x' = V^*x$,

$$b = Ax \iff b' = \Sigma x'.$$

- rank(A) is the number of non-zero singular values.
- range(A) $=< u_1, \ldots, u_r >$
- null(A) $=< v_{r+1}, \ldots, v_n >$

The singular values of A are the square roots of the eigenvalues of A^*A. The eigenvectors of A^*A are the columns of V. The eigenvectors of AA^* are the columns of U.

4.3.2 Low-Rank Approximation

1. $A = \sum_{j=1}^{r} \sigma_j u_j v_j^*$
2. $||A - \sum_{j=1}^{k} \sigma_j u_j v_j^*||_2 = \sigma_{k+1}$

σ_{k+1} is the smallest 2-norm distance between A and a matrix of rank k.

4.4 Projectors

A **projector** is a matrix P satisfying $P^2 = P$. The eigenvalues and singular values of P are 1 or 0.

The **complementary projector** to P is $I - P$

- range(P) = null($I - P$)
- range($I - P$) = null(P)
- null(P) $\cap$ null($I - P$) = $\{0\}$

A projector P satisfying $P^* = P$ is an **orthogonal projector**. range(P)$\perp$null(P).

Projection onto a vector: For q normal ($q^*q = 1$), $P_q = qq^*$ is the projection onto q. $P_{\perp q} = I - qq^*$ is the projection onto the space orthogonal to q. For arbitrary vectors, $P_a = \frac{aa^*}{a^*a}$ is the projection onto a. $P_{\perp a} = I - \frac{aa^*}{a^*a}$ is the projection onto the space orthogonal to a.

Projection onto a subspace: If $\{a_1, \ldots, a_k\}$ is a basis for $S \subset \mathbb{C}^m$, the orthogonal projector onto S is

$$P = A(A^*A)^{-1}A^* \text{ with } A = (a_1 \ldots a_k)$$

If $\{q_1, \ldots, q_k\}$ is an orthonormal basis for $S \subset \mathbb{C}^m$, the orthogonal projector onto S is

$$P = \hat{Q}\hat{Q}^* \text{ with } \hat{Q} = (q_1 \ldots q_k)$$

4.5 Householder Triangularization

Gram-Schmidt is a form of **triangular orthogonalization**

$$\begin{aligned} AR_1 \ldots R_n &= Q,\ R = (R_1 \ldots R_n)^{-1} \\ R_j &= \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \frac{1}{\sigma_{jj}} & -\frac{q_{j,j+1}}{\sigma_{jj}} & \ldots \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{aligned}$$

Householder triangularization, however, is **orthogonal triangularization**,

$$\begin{aligned} Q_n \ldots Q_1 A &= R,\ Q = (Q_n \ldots Q_1)^* \\ Q_j &= \begin{pmatrix} I & 0 \\ 0 & F_j \end{pmatrix} \end{aligned}$$

where $F_k = I - 2\frac{v_k v_k^*}{v_k^* v_k}$. For $v_k = \text{sgn}(x_1)||x||e_1 + x$, $x = A_{k:m,k}$. F_k is a reflection of $\mathbb{C}^{m-k+1}$ across the hyperplane perpendicular to the vector connecting $||x||e_k$ and x.

Algorithm for $k = 1 : n$

1. $x = A_{k:m,k}$
2. $v_k = \text{sgn}(x_1)||x||_2 e_1 + x$
3. $v_k = \frac{v_k}{||v_k||_2}$
4. $A_{k:m,k:n} = A_{k:m,k:n} - 2v_k(v_k * A_{k:m,k:n})$

Calculating Q^*b

for $k = 1 : n$

$$b_{k:m} = b_{k:m} - 2v_k(v_k^* b_{k:m})$$

Calculating Qx

for $k = n : -1 : 1$

$$x_{k:m} = x_{k:m} - 2v_k(v_k^* x_{k:m})$$

4.6 Condition Numbers

Absolute

$$\hat{K} = \sup_{\delta x} \frac{||\delta f||}{||\delta x||}$$

Relative

$$K = \sup_{\delta x} \left(\frac{||\delta f||}{||\delta x||} \Big/ \frac{||f||}{||x||} \right)$$

$Ax = b$

- $K_{x \mapsto b} = \frac{||A||||x||}{||Ax||} \leq ||A||||A^{-1}||$
- $K_{b \mapsto x} = \frac{||A^{-1}||||b||}{||A^{-1}b||} \leq ||A||||A^{-1}||$
- $K_{A \mapsto x,b} = ||A||||A^{-1}|| = \kappa(A)$

The above κ is typically denoted as the **condition number** of a matrix A.

4.7 Floating Point Arithmetic

$$\mathbb{F} = \left\{ x | x = \pm\left(\frac{m}{B^t}\right)B^e,\ m, B \in \mathbb{Z} \right\} \cup \{0\}$$

Let $\epsilon_{\text{machine}} = \frac{1}{2}B^{1-t} \equiv \epsilon_m$

1. $\forall x \in \mathbb{R}$, $\exists \epsilon$ with $|\epsilon| \leq \epsilon_m$, with $fl(x) = x(1+\epsilon)$.
2. $\forall x, y \in \mathbb{F}$, $\exists \epsilon$, $|\epsilon| \leq \epsilon_m$, with $x \otimes y = x \times y(1+\epsilon)$

4.7.1 Stability

Accuracy: An algorithm $\tilde{f}$ for a problem f is **accurate** if $\forall x \in X$,

$$\frac{||\tilde{f}(x) - f(x)||}{||f(x)||} = \mathcal{O}(\epsilon_m)$$

Stability: An algorithm $\tilde{f}$ for a problem f is **stable** if $\forall x \in X$,

$$\frac{||\tilde{f}(x) - f(\tilde{x})||}{||f(\tilde{x})||}$$

for some $\tilde{x}$ with

$$\frac{||x-\tilde{x}||}{||x||} = \mathcal{O}(\epsilon_m)$$

Backwards Stability: An algorithm $\tilde{f}$ for a problem f is **backwards stable** if $\forall x \in A,\ \tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ with

$$\frac{||x-\tilde{x}||}{||x||} = \mathcal{O}(\epsilon_m)$$

Theorem. *If a backwards stable algorithm $\tilde{f}$ for a problem f with condition number K is implemented on a machine employing floating-point arithmetic,*

$$\frac{||\tilde{f}(x) - f(x)||}{||f(x)||} = \mathcal{O}(K(x)\epsilon_m)$$

4.8 Least-Squares

The **least-squares** problem is to find the best approximation to x for $Ax = b$, $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$, i.e. to minimize $r = b - Ax$.

Theorem. *$x \in C^n$ minimizes $||r||_2 = ||b - Ax||_2$*

$$\begin{aligned} &\iff r \perp range(A) \\ &\iff A^*r = 0 \\ &\iff A^*Ax = A^*b \\ &\iff Ax = Pb, \end{aligned}$$

where P is the projector onto range(A).

x is unique iff A is invertible. Otherwise, $x + y$, $y \in null(A)$ is also a solution

If A is full rank, $x = (A^*A)^{-1}A^*b = A^+b$, with A^+ known as the **pseudoinverse**. Then $AA^+ = A(A^*A)^{-1}A^* = P$, so $Ax = Pb$.

To solve least-squares, solve either $x = A^+b$, or $y = Pb$.

4.8.1 Backwards Stability of Least-Squares via QR-Factorization

1. If $\tilde{Q}\tilde{R} = \text{qr}(A)$, then (1) $\tilde{Q}\tilde{R} = A + \delta A$, with (2) $\frac{||\delta A||}{||A||} = \mathcal{O}(\epsilon_m)$.
2. If $\tilde{y} = Q \otimes b$, then (1) $\tilde{y} = (Q + \delta Q) \times b$, with (2) $||\delta Q|| = \mathcal{O}(\epsilon_m)$.
3. If $\tilde{x} = R\backslash b$, then (1) $\tilde{x} = (R + \delta)^{-1}b$, with (2) $\frac{||\delta R||}{||R||} = \mathcal{O}(\epsilon_m)$.

4.8.2 Stability of Back-Substitution

If $\tilde{x} = R\backslash b$, $\tilde{x} = (R + \delta R)^{-1}b$ with $\frac{||\delta R||}{||R||} = \mathcal{O}(\epsilon_m)$,

$$(d_{ij}) = \frac{|\delta r_{ij}|}{|r_{ij}|} \le m\epsilon_m + \mathcal{O}(\epsilon_m^2), \text{ and}$$

$$\left(d_{ij}\right) \le \begin{pmatrix} m & 1 & \dots & m-1 \\ & m-1 & \dots & m-2 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \epsilon_m + \mathcal{O}(\epsilon_m^2)$$

4.8.3 Conditioning of Least-Squares

For $A \in \mathbb{C}^{m\times n}$, full rank, $b \in \mathbb{C}^m$, $A^+ = (A^*A)^{-1}A^*$, $x = A^+b$ is the point in $\mathbb{C}^n$ closest to b, $y = Ax$ is the projection of b onto range(A).

$$\begin{aligned} K(A) &= ||A||||A^+|| = \frac{\sigma_1}{\sigma_n},\ t \le K \le \infty \\ \theta &= \cos^{-1}\left(\frac{||y||}{||b||}\right),\ 0 \le \theta \le \frac{\pi}{2} \\ \eta &= \frac{||A||||x||}{||Ax||},\ 1 \le \eta \le K. \end{aligned}$$

Theorem. *Given $b \in \mathbb{C}^m$, $A \in \mathbb{C}^{m\times n}$, full-rank has 2-norm relative condition numbers*

	y	x
b	$\frac{1}{\cos\theta}$	$\frac{K(A)}{\eta\cos\theta}$
A	$\frac{K(A)}{\cos\theta}$	$K(A) + \frac{K(A)^2\tan\theta}{\eta}$

4.8.4 Stability of Least-Squares

- **Full Rank**

  Stable

  Solution of normal equations, when restricted to problems where $K(A)$ is uniformly bounded above or $\tan\theta/\eta$ is uniformly bounded below.

- **Backwards Stable**

  QR-factorization by Gram-Schmidt provided $\hat{Q}^*b$ is computed implicitly.

  QR-factorization by Householder.

  SVD

- **Rank-Deficient**

  Unique solution if $||x||$ minimized; solution depends upon rank. SVD is the only stable algorithm.

4.9 Gaussian Elimination

Amounts to solving $LA = U$. This algorithm is unstable for solving linear systems, since L and U may move factors of arbitrary size; where L is unit lower-triangular.

4.10 Eigenvalues and Eigenvectors

Eigenvalue Decomposition: If A is square and nonsingular, $A = X\Lambda X^{-1}$, where the columns X are eigenvectors of A and $\text{diag}(\Lambda) = \text{spec}(A)$, so $Ax_j = \lambda_j x_j$.

Change of Basis: For $A = X\Lambda X^{-1}$, if $Ax = b$, then $\Lambda(X^{-1}x) = X^{-1}b$.

For $\lambda \in \text{spec}(A)$, the **eigenspace** E_λ is the space spanned by the eigenvectors of A corresponding to λ.

E_λ is also the null space of $A - \lambda I$.

E_λ is invariant under A: $AE_\lambda \subseteq E_\lambda$.

4.10.1 Characteristic Polynomial

For $A \in \mathbb{C}^{m\times m}$, $p_A(z) = \det(zI - A)$.

$\lambda \in \text{spec}(A) \iff p_A(\lambda) = 0$.

4.10.2 Multiplicity

1. Geometric multiplicity: $\dim(E_\lambda)$.
2. Algebraic multiplicity: the multiplicity of λ as a root of $p_A(z)$.

Theorem. *If $A \in \mathbb{C}^{m\times m}$, then A has m eigenvalues, counted with algebraic multiplicity.*

Theorem. *The algebraic multiplicity of $\lambda \in spec(A)$ is greater than or equal to its geometric multiplicity.*

4.10.3 Similarity Transformations

If $X \in \mathbb{C}^{m\times m}$ is nonsingular, the map $S_x : \mathbb{C}^{m\times m} \to \mathbb{C}^{m\times m}$, $A \mapsto XAX^{-1}$ is called a similarity transformation. A is similar to B if $A = S_x(B)$ for some $X \in \mathbb{C}^{m\times m}$.

Theorem. *For X nonsingular, A and XAX^{-1} have the same characteristic polynomial, and their eigenvalues have the same geometric multiplicity.*

4.10.4 Defective Eigenvalues

Defective eigenvalues: An eigenvalue whose algebraic multiplicity exceeds its geometric multiplicity.

Defective matrix: A matrix with at least one defective eigenvalue.

Theorem. *If a matrix is nondefective, then it is nonsingular (invertible).*

4.10.5 Determinants and Trace

1. $\det(A) = \prod_{i=1}^n \lambda_i$
2. $\text{tr}(A) = \sum_{i=1}^n \lambda_i$.

4.10.6 Factorizations

1. A diagonalization $A = X\Lambda X^{-1}$ exists iff A is nondefective.
2. A unitary diagonalization $A = Q\Lambda Q^*$ exists iff A is normal.
3. **Schur Factorization**: $A = QTQ^*$, where Q is unitary, T is upper-triangular, exists for all square matrices.

4.11 Interpolation and Integration

4.11.1 Lagrange Interpolation

Given a function f and $x_0 < \cdots < x_n \in \mathbb{R}$,

$$\begin{aligned} P(x; x_j, f(x_j)) &= \sum_{j=0}^{n} f(x_j)L_j(x), \text{ where} \\ L_j(x) &= \frac{\prod_{k\neq j}(x - x_k)}{(x_j - x_k)}, \end{aligned}$$

is the unique polynomial of degree $\leq n$ with $P(x_j) = f(x_j)$.

4.11.2 Numerical Integration

Given f integrable on $[a,b]$, and $a = x_0 < \cdots < x_n = b$,

$$\begin{aligned}\int_a^b f(x)dx &= \sum_{j=0}^{n} c_j f(x_j) + E, \text{ where} \\ c_j &= \int_a^b L_j(x)dx \\ |E| &\leq \frac{M}{(n+1)!}\int_a^b \prod_{j=0}^{n} |x - x_j| dx \\ M &= \sup_{[a,b]} |f^{n+1}(x)|\end{aligned}$$

4.11.3 Trapezoid Rule

$$\begin{aligned}\int_a^b f(x)dx &= h\left(\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2}\right) + E \\ |E| &\leq \frac{Mh^2}{12}(b-a) \\ M &= \sup_{[a,b]} |f''(x)|\end{aligned}$$

4.12 Stability of ODES

Stability: For the ODE system $y' = f(t,y),\ t \geq 0$, a solution $u(t)$ is

- **Stable** if $\forall \epsilon > 0$, $\exists \delta > 0$ with $\forall \hat{u}, u$ solving the ODE, $|u(t) - \hat{u}(t)| \leq \epsilon,\ \forall t \geq 0$ when $|u(0) - \hat{u}(0)| \leq \delta$.
- **Asymptotically Stable** if $u(t)$ is stable and $\lim_{t\to\infty} |u(t) - \hat{u}(t)| = 0$.

Test Equation ($y' = \lambda y$)

All trajectories are stable if $\text{Re}(\lambda) \leq 0$.

All trajectories are asymptotically stable if $\text{Re}(\lambda) < 0$.

4.12.1 Linear, Constant Coefficient

Stable:

Nondefective: if $\text{Re}(\lambda_i) \leq 0\ \forall i$.

Defective: either $\text{Re}(\lambda) < 0$ or λ is simple and $\text{Re}(\lambda) = 0$.

Asymptotically Stable:

If $\text{Re}(\lambda_i) < 0\ \forall i$.

4.12.2 Linear, Variable-Coefficient

$$y' = A(t)y + q(t)$$

which has fundamental solution $Y(t)$, solves

$$\begin{aligned}Y'(t) &= A(t)Y(t) \\ Y(0) &= I.\end{aligned}$$

General solution, with $y(0) = c$:

$$y(t) = Y(t)\left[c + \int_0^t Y^{-1}(s)q(s)ds\right]$$

Stable: if $\sup_{0\leq t<\infty} ||Y(t)|| < \infty$.

Asymptotically stable: if $\lim_{t\to\infty} ||Y(t)|| = 0$

Stability constant: $K = \sup_{0\leq s\leq t<\infty} ||Y(t)Y^{-1}(s)||$.

4.12.3 Nonlinear

$$y' = f(y)$$

Can apply linear stability analysis to

$$z' = A(t,y)z = \left(\frac{\partial f}{\partial y}\right)_t z.$$

The eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial y}$ vary with t, so a trajectory may alternate between stability and instability.

4.13 Consistency, Stability, Convergence

For a numerical method given by the equation $\mathcal{N}_h u(t_n) = \mathcal{O}$,

4.13.1 Consistency

The method is consistent of order p if the local truncation error,

$$d_n = \mathcal{N}_h y(t_n) = \mathcal{O}(h^p)$$

where y is the exact solution, and $h = \max h_n$.

4.13.2 Convergence

The method is convergent of order p if the global error

$$e_n = y_n - y(t_n) = \mathcal{O}(h^p)$$

where y_n is the numerical solution, y is the exact solution, and $h = \max h_n$.

4.13.3 0-Stability

The difference operator $\mathcal{N}_h u(t_n)$ is 0-stable if $\exists h_0, K$ with

$$|x_n - z_n| \leq \left[|x_0 - z_0| + \max_{1\leq j\leq N} |\mathcal{N}_h x(t_j) - \mathcal{N}_j z(t_j)|\right] K$$

$\forall h < h_0$, $\forall$ mesh functions x_n, z_n.

Theorem. *Consistency and 0-stability $\implies$ Convergence*

4.13.4 Absolute Stability

Region of Absolute Stability: The region of the complex plane A s.t. when $h\lambda \in A$, and the numerical method is applied to $y' = \lambda y$ with stepsize h, $\left|\frac{y_n}{y_{n-1}}\right| \leq 1$ for the calculated solution $\{y_n\}$.

System with Constant Coefficients: Solve $y' = Ay$ with a numerical method. $\{h\lambda_1, \ldots, h\lambda_n\}$ all lie within the region of absolute stability for the method, then $|y_n| \leq \text{cond}(x)|y_0|$, where $A = X\Lambda X^{-1}$.

Stiffness: The test equation, if solved on $(0, b)$ is **stiff** if $b\text{Re}(\lambda) << -1$. An ODE $y' = f(y)$, $\{\lambda_j\} = \text{spec}\left(\frac{\partial f}{\partial y}\right)$, is **stiff** if $b \min_j \text{Re}(\lambda_j) << -1$.

A-Stability: A numerical method is A-stable if its region of absolute stability includes the entire left half-plane.

4.14 Numerical Methods for ODES

$z = h\lambda$.

4.14.1 Forward Euler

$$\mathcal{N}_h y_n = \frac{y_n - y_{n-1}}{h_n} - f(t_{n-1}, y_{n-1}) = \mathcal{O}$$

Consistent and convergent of order 1 with region of absolute stability: $|z + 1| \leq 1$.

4.14.2 Backward Euler

$$\mathcal{N}_h y_n = \frac{y_n - y_{n-1}}{h_n} - f(t_n, y_n) = \mathcal{O}$$

Consistent and convergent of order 1. It is A-stable; the region of absolute stability is $|z - 1| \geq 1$. Stiff decay.

4.14.3 Trapezoidal

$$\mathcal{N}_h y_n = \frac{y_n - y_{n-1}}{h_n} - \frac{1}{2}\Big[f(t_n, y_n) + f(t_{n-1}, y_{n-1})\Big] = \mathcal{O}$$

Consistent and convergent of order 2. It is A-stable; the region of absolute stability is $\text{Re}(z) < 0$.

4.14.4 Implicit Midpoint

$$\mathcal{N}_h y_n = \frac{y_n - y_{n-1}}{h_n} - f\left(t_{n-\frac{1}{2}}, \frac{1}{2}(y_n + y_{n-1})\right) = \mathcal{O}$$

Consistent and convergent of order 2. A-stable; the region of absolute stability: $\text{Re}(z) < 0$.

Stiff decay: A numerical method has stiff decay if, when applied to the stiff test equation $y' = \lambda(y - g(t))$ with stepsize h,

$$\lim_{\text{Re}(h\lambda)\to -\infty} |y_n - g(t_n)| = 0$$

4.14.5 Taylor Methods

The Taylor method of order $p + 1$ is

$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) + \ldots \frac{h^p}{p!}\frac{\partial^p}{\partial t^p} f(t_{n-1}, y_{n-1})$$

The region of A-stability is

$$\left\{z : \left|1 + z + \ldots \frac{z^p}{p!}\right| \leq 1\right\}$$

4.14.6 Runge-Kutta Methods

A general s-stage Runge-Kutta method is specified by $A \in \mathbb{C}^{s\times s}$, $b \in \mathbb{C}^s$, s.t.

$$\begin{aligned} Y_i &= y_{n-1} + h\sum_{j=1}^{s} a_{ij} f(t_{n-1} + c_j h, Y_j), i = 1, \ldots, s \\ y_n &= y_{n-1} + h\sum_{j=1}^{s} b_j f(t_{n-1} + c_j h, Y_j), \\ c_i &= \sum_{j=1}^{s} a_{ij} \end{aligned}$$

The c_i's are the fractional times that each Y_i is an approximation to.

Order Conditions: For a Runge-Kutta method to be of order p,

$$b^T A^k C^{l-1}\mathbf{1} = \frac{(l-1)!}{(k-1)!}$$

for $1 \le l \le p$, $0 \le k \le p-l$. An explicit RK method can have at most order s.

Absolute Stability: For an RK method of order p,

$$\frac{y_n}{y_{n-1}} = R(z) = 1 + z + \dots \frac{z^p}{p!} + \sum_{j>p} z^j b^T A^{j-1} \mathbf{1}$$

The region of absolute stability is given by $\{z : |R(z)| \le 1\}$. Since $R(z)$ is a polynomial, no explicit RK method can be A-stable.

Every explicit, p-stage method of order p has the same region of absolute stability:

$$\{z : |1 + z + \dots \frac{z^p}{p!}| \le 1\}$$

4.14.7 Finite Element Methods

4.14.8 Multistep Linear Methods

The general k-step MSL method is specified by the formula

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = \sum_{j=0}^{k} \beta_j f_{n-j},$$

where $\alpha_0 = 0$, $|\alpha_k| + |\beta_k| \ne 0$. If $\beta_0 = 0$ is explicit. If $\beta_0 \ne 0$, the method is implicit.

Order Conditions For the k-step MSL above, let

$$\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j \\ C_i &= (-1)^i \left[\frac{1}{i!} \sum_{j=1}^{k} j^i \alpha_j + \frac{1}{(i-1)!} \sum_{j=0}^{k} j^{(i-1)} \beta_j \right] \end{aligned}$$

for $i \ge 1$.

Then the MSL has order p iff $C_0 = C_1 = \dots = C_p = 0$, $C_{p+1} \ne 1$.

Consistency: The k-step MSL is **consistent** if its order is ≥ 1, i.e. if

$$\begin{aligned} \sum_{j=0}^{k} \alpha_j &= 0 \\ \sum_{j=0}^{k} j\alpha_j + \sum_{j=0}^{k} \beta_j &= 0 \end{aligned}$$

Stability for MSL Methods

Difference equations: For the difference equation

$$a_k y_{n-k} + \dots + a_0 y_n = q_n,$$

the solution has the form

$$y_n = \sum_{i=0}^{k} c_i \xi_i^n$$

where ξ_i are the roots of the associated characteristic equation, $\Phi(\xi) = \sum_{i=0}^{k} a_i \xi^{k-i}$.

This difference equation is **stable** if $|\xi_i| \le 1 \ \forall i$; it is **asymptotically stable** if $|\xi_i| < 1 \ \forall i$.

0-Stability: The k-step MSL is 0-stable iff all roots ξ_i of the characteristic polynomial $\rho(\xi)$, satisfy

$$|\xi_i| \le 1,$$

and if $|\xi_i| = 1$, then ξ_i is a simple root, $1 \le i \le k$.

Strongly stable: MSL method is **strongly stable** if all roots of $\rho(\xi) = 0$ are inside the unit circle except for the root $\xi = 1$.

Weakly stable: MSL method is **weakly stable** if it is 0-stable but not strongly stable.

Absolute Stability: The boundary of the region of absolute stability is

$$B(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, \ 0 \le \theta \le 2\pi.$$

A few more things: An explicit MSL method cannot be A-stable. An A-stable MSL method can have order at most 2.

4.14.9 BDF Methods

MSL methods based on backwards difference approximations to the derivative; designed for stiff decay with

$$\begin{aligned} \alpha_j &= \frac{\gamma_j}{\gamma_0}; \ \beta_0 = \frac{1}{\gamma_0}, \ \beta_j = 0 \\ \gamma_j &= -\frac{\prod_{i \ne j}(i-u)}{\prod_{i \ne j}(i-j)}|_{u=0} \end{aligned}$$

Order k; maximal 0-stable order is 6. A-stable with stiff-decay.

4.14.10 Shooting

For BVP $y' = f(t,y)$, $g(y(0), y(b)) = 0$, let $y(t;c)$ be a solution with $y(0) = c$, $h(c) = g(c, y(b;c)) = 0$.

Solve above equation for c using Newton's method: $c^{\nu+1} = c^{\nu} + Q^{-1} h(c^{\nu})$.

Algorithm

while $|c^{\nu} - c^{\nu-1}| > \text{Tol}$

1. Solve $y' = f(t, y)$, $y(0) = c^\nu$.
2. Construct $h(c^\nu) = g(c^\nu, y^\nu(b))$.
3. Solve $Y' = A(t, y^\nu)Y$, $Y(0) = I$, using $A(t, y^\nu) = \frac{\partial f}{\partial y}(t, y^\nu)$.
4. Form $Q = B_0 + B_b Y^\nu(b)$.
5. $c^{\nu+1} = c^\nu + Q^{-1}h(c^\nu)$.

Stability depends on stability of the IVP. Order depends on the order of the IVP solver.

5 Methods of Applied Mathematics

5.1 Fourier Analysis

Define the Fourier transform and Inverse Fourier transform

$$\begin{aligned} \hat{f}(k) &= \int_{\mathbb{R}} f(x)e^{ikx}dx \\ f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k)e^{-ikx}dk \end{aligned}$$

Some useful properties of the FT

- $\hat{\hat{f}}(x) = 2\pi f(-x)$
- $\widehat{f(x-b)} = e^{ikb}\hat{f}(k)$
- $\widehat{f(ax)} = \frac{1}{|a|}\hat{f}\left(\frac{k}{a}\right)$
- $\widehat{f(ax-b)} = \frac{1}{|a|}e^{\frac{ikb}{a}}\hat{f}\left(\frac{k}{a}\right)$
- $\widehat{\frac{d^n}{dx^n}f(x)} = (-ik)^n\hat{f}(k)$
- $\widehat{(ix)^n f(x)} = \frac{d^n}{dk^n}\hat{f}(k)$
- $(f, g) = \frac{1}{2\pi}(\hat{f}, \hat{g}) \implies ||f||_2^2 = \frac{1}{2\pi}||\hat{f}||_2^2$

5.1.1 Convolution Theorem

$$\begin{aligned} \widehat{f*g}(k) &= \hat{f}(k)\hat{g}(k) \\ \text{where } f*g(x) &= \int f(y)g(x-y)dy \end{aligned}$$

5.1.2 Riemann-Lebesgue Lemma

If $f \in L^1(\mathbb{R})$, then $\hat{f}(k) \to 0$ as $k \to \infty$.

5.1.3 Other Useful Fourier Transform Stuff

- $f \in L^1 \implies \hat{f}$ is uniformly continuous.
- $f \in L^2 \implies \hat{f} \in L^2$.
- For $1 < p \le q < \infty$, $f \in L^p \implies \hat{f} \in L^q$, for $\frac{1}{p} + \frac{1}{q} = 1$.
- $f \in C^n \implies |k^n\hat{f}(k)| \to 0$ as $k \to \infty$.

5.1.4 Explicit FT Evaluation

- $\widehat{e^{-a|x|}} = \frac{2a}{a^2+k^2}$
- $\widehat{\frac{1}{x^2+a^2}} = \frac{\pi}{a}e^{-a|k|}$
- $\widehat{\text{sech}(x)} = \pi\text{sech}\left(\frac{k\pi}{2}\right)$
- $\widehat{e^{-ax^2}} = \sqrt{\frac{\pi}{a}}e^{-\frac{k^2}{4a}}$
- Let

$$f(x) = \begin{cases} b & |x| < a \\ 0 & |x| > a \end{cases}$$

$$\hat{f}(k) = \frac{2b\sin(ka)}{k}$$

- $\widehat{f(x)g(x)} = \frac{1}{2\pi}\hat{f}*\hat{g}$
- $\widehat{f(x)e^{i\omega x}} = \hat{f}(k+w)$

via distribution, the following can be calculated

1. $\hat{\delta}(k) = 1$
2. $\hat{1}(k) = 2\pi\delta(k)$
3. $\hat{H}(k) = \pi\delta(k) + i\text{PV}\frac{1}{k}$
4. $\widehat{\text{sgn}}(k) = 2\hat{H}(k) - \hat{1}(k) = 2i\text{PV}\frac{1}{k}$
5. $<\hat{T}, \hat{\varphi}> = 2\pi <T, \varphi>$

5.2 Fourier Series

Trigonometric formulation, on an interval (a, b),

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{b-a}\right) \\ &+ b_n \sin\left(\frac{2\pi nx}{b-a}\right) \\ a_n &= \frac{2}{b-a} \int_0^{b-a} f(x) \cos\left(\frac{2\pi nx}{b-a}\right) dx \\ b_n &= \frac{2}{b-a} \int_0^{b-a} f(x) \sin\left(\frac{2\pi nx}{b-a}\right) dx \end{aligned}$$

Exponential formulation

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi nix}{b-a}} \\ c_n &= \frac{1}{b-a} \int_0^{b-a} f(x) e^{\frac{2\pi nix}{b-a}} \end{aligned}$$

These notes are incomplete